Gradient, Divergence and Curl in orthogonal curvilinear Coordinates; —

* First, I review here some impt. point.

We have studied in last note that square of the length of displacement ($\vec{ds}$) is given by

$$\boxed{ds^2 = \sum_{i=1}^{3}\sum_{j=1}^{3} g_{ij}\, du_i\, du_j} \quad \text{— ①}$$

~~$ds^2 = du_1 du_2 du_3$~~

where g_{ij} are metric coefficients.

$$\boxed{g_{ij} = \frac{\partial \vec{r}}{\partial u_i} \cdot \frac{\partial \vec{r}}{\partial u_j}} \quad \text{— ②}$$

$\frac{\partial \vec{r}}{\partial u_i} \rightarrow$ tangent vector to the curve $\vec{r}$ for $u_i =$ const.

For orthogonal coordinate system

$$g_{ij} = 0, \text{ for } i \neq j$$

and defining $g_{ii} = h_i^2$, we write

$$\boxed{\begin{aligned} ds^2 &= (h_1 du_1)^2 + (h_2 du_2)^2 + (h_3 du_3)^2 \\ &= \sum_{i=1}^{3} (h_i du_i)^2 \end{aligned}} \quad \text{— ③}$$

where h_1, h_2, h_3 are scale factors

~~Again~~ we i.

Next, we identify (from ③)

$$\boxed{\begin{aligned} ds_i &= h_i du_i \\ \frac{\partial \vec{r}}{\partial u_i} &= h_i \hat{u}_i = h_i \hat{e}_i \end{aligned}} \quad \text{— ④}$$

$h_i du_i \rightarrow$ have dimension of length.

Here $\hat{u}_i$ or $\hat{e}_i$, ~~are~~ we take same unite vectors.

$ds_i \rightarrow$ differential length along the direction $\hat{u}_i$ (or $\hat{e}_i$).

→ For Cartesian coordinates.

$$g_{ii} = 1, \text{ i.e., } g_{11} = g_{22} = g_{33} = 1.$$

→ Area and volume elements are obtained as.

$$\boxed{d\sigma_{ij} = ds_i ds_j = h_i h_j du_i du_j} \quad \text{— (5)}$$

→ Area element

Volume element

$$\boxed{d\tau = ds_1 ds_2 ds_3 = h_1 h_2 h_3 du_1 du_2 du_3} \quad \text{— (6)}$$

Jacobian :- To define Jacobian, we consider Cartesian surface element $dx dy$.

Note :- * For transforming one coordinate system to other, we use this tool.

* How to choose a particular coordinate system? We look at the symmetry of the particular problem

e.g., spherical, cylindrical, paraboloid. etc

The infinitesimal rectangle in the new coordinates u_1, u_2 (earlier it was x, y) formed by two incremental vectors $d\vec{r_1}, d\vec{r_2}$

~~[illegible]~~

$$d\vec{r_1} = \vec{r}(u_1 + du_1, u_2) - \vec{r}(u_1, u_2) = \frac{\partial \vec{r}}{\partial u_1} du_1,$$

$$d\vec{r_2} = \vec{r}(u_1, u_2 + du_2) - \vec{r}(u_1, u_2) = \frac{\partial \vec{r}}{\partial u_2} du_2$$

~~New area dxdy gi~~

Area element $dx dy$ given by (z-component of their cross product)

$$dx dy = d\vec{r_1} \times d\vec{r_2} \Big|_{z\text{-component}}$$

$$dxdy = d\vec{r}_1 \times d\vec{r}_2 \Big|_z = \begin{bmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial x}{\partial u_1} du_1 & \frac{\partial y}{\partial u_1} du_1 & \frac{\partial z}{\partial u_1} du_1 \\ \frac{\partial x}{\partial u_2} du_2 & \frac{\partial y}{\partial u_2} du_2 & \frac{\partial z}{\partial u_2} du_2 \end{bmatrix}_z$$

$$= \cancel{\hat{x}\ \text{Co}}$$

Take only z component

$$dxdy = \left(\frac{\partial x}{\partial u_1} \frac{\partial y}{\partial u_2} - \frac{\partial x}{\partial u_2} \frac{\partial y}{\partial u_1} \right) du_1 du_2$$

or
$$\boxed{dxdy = \underbrace{\begin{vmatrix} \frac{\partial x}{\partial u_1} & \frac{\partial x}{\partial u_2} \\ \frac{\partial y}{\partial u_1} & \frac{\partial y}{\partial u_2} \end{vmatrix}} du_1 du_2} \quad \text{— (7)}$$

or
$$\boxed{dxdy = J \, du_1 du_2} \qquad (= h_1 \cdot h_2)$$

J is called Jacobian and given by $\begin{vmatrix} \frac{\partial x}{\partial u_1} & \frac{\partial x}{\partial u_2} \\ \frac{\partial y}{\partial u_1} & \frac{\partial y}{\partial u_2} \end{vmatrix}$ for the above transformation.

~~For Volume~~

For Volume element $dx\,dy\,dz$, the transformation to coordinates (u_1, u_2, u_3) is ~~are~~ obtained as (use similar procedure)

$$\boxed{dx\,dy\,dz = \underbrace{\begin{vmatrix} \frac{\partial x}{\partial u_1} & \frac{\partial x}{\partial u_2} & \frac{\partial x}{\partial u_3} \\ \frac{\partial y}{\partial u_1} & \frac{\partial y}{\partial u_2} & \frac{\partial y}{\partial u_3} \\ \frac{\partial z}{\partial u_1} & \frac{\partial z}{\partial u_2} & \frac{\partial z}{\partial u_3} \end{vmatrix}}_{h_1 h_2 h_3} du_1\,du_2\,du_3} \quad \text{--- (8)}$$

Homework – obtain Jacobian for polar coordinates, spherical coordinates, cylindrical coordinates. Note that you are taking transformation of the Cartesian coordinate to the above mentioned coordinates.

Gradient:

Gradient of scalar function $\psi(u_1, u_2, u_3)$, i.e. $\nabla\psi$, in the direction normal to $u_1 =$ const. is given by

$$\hat{e}_1 \cdot \nabla\psi = \frac{\partial \psi}{\partial s_1} = \frac{1}{h_1}\frac{\partial \psi}{\partial u_1}$$

{Here we have fixed, q_2, q_3; q_1 varies}

~~For general~~ Therefore, we can write

$$\nabla\psi = \hat{e}_1 \frac{\partial \psi}{\partial s_1} + \hat{e}_2 \frac{\partial \psi}{\partial s_2} + \hat{e}_3 \frac{\partial \psi}{\partial s_3}$$

or

$$\boxed{\nabla\psi = \frac{\hat{e}_1}{h_1}\frac{\partial \psi}{\partial u_1} + \frac{\hat{e}_2}{h_2}\frac{\partial \psi}{\partial u_2} + \frac{\hat{e}_3}{h_3}\frac{\partial \psi}{\partial u_3}} \quad \text{--- (9)}$$

From this

$$\nabla \equiv \frac{\hat{e}_1}{h_1}\frac{\partial}{\partial u_1} + \frac{\hat{e}_2}{h_2}\frac{\partial}{\partial u_2} + \frac{\hat{e}_3}{h_3}\frac{\partial}{\partial u_3}$$

Divergence and Curl :-

Take an arbitrary vector $\vec{V}$. defined by

$$\vec{V} = \hat{e}_1 V_1 + \hat{e}_2 V_2 + \hat{e}_3 V_3$$

V_1, V_2, V_3 are its components.

We can write

$$\vec{V} = \frac{\hat{e}_1}{h_2 h_3}(h_2 h_3 V_1) + \frac{\hat{e}_2}{h_1 h_3}(h_1 h_3 V_2) + \frac{\hat{e}_3}{h_1 h_2}(h_1 h_2 V_3)$$

Now the divergence $\nabla\cdot\vec{V}$ is given by

$$\nabla\cdot\vec{V} = \frac{\hat{e}_1}{h_2 h_3}\cdot\nabla(h_2 h_3 V_1) + \frac{\hat{e}_2}{h_1 h_3}\cdot\nabla(h_1 h_3 V_2) + \frac{\hat{e}_3}{h_1 h_2}\cdot\nabla(h_1 h_2 V_3)$$

or

$$\boxed{\nabla\cdot\vec{V} = \frac{1}{h_1 h_2 h_3}\left[\frac{\partial}{\partial u_1}(h_2 h_3 V_1) + \frac{\partial}{\partial u_2}(h_1 h_3 V_2) + \frac{\partial}{\partial u_3}(h_1 h_2 V_3)\right]} \quad (1)$$

Note: Here we have used the identity

$$\nabla\cdot(\phi\vec{A}) = \vec{A}\cdot\nabla\phi + \phi\nabla\cdot\vec{A}.$$

Take only component (1).

$$\left\{\vec{V}_1 = \frac{\hat{e}_1}{h_2 h_3}(h_2 h_3 V_1)\right\}$$

$$\nabla\cdot\vec{V}_1 = \nabla\cdot\left[\underbrace{\frac{\hat{e}_1}{h_2 h_3}}_{A_1}\underbrace{(h_2 h_3 V_1)}_{\phi}\right]$$

$$= \frac{\hat{e}_1}{h_2 h_3}\cdot\nabla(h_2 h_3 V_1) + (h_2 h_3 V_1)\nabla\cdot\left(\frac{\hat{e}_1}{h_2 h_3}\right)$$

$$= \frac{\hat{e}_1}{h_2 h_3}\cdot\left\{\frac{\hat{e}_1}{h_1}\frac{\partial(h_2 h_3 V_1)}{\partial u_1} + \cancel{\hat{e}_2(\ )}^{\,0} + \cancel{\hat{e}_3(\ )}^{\,0}\right\} + 0$$

$$\boxed{\nabla\cdot\vec{V}_1 = \frac{1}{h_1 h_2 h_3}\frac{\partial(h_2 h_3 V_1)}{\partial u_1}}$$

For other two components use same procedure

Curl —

$\nabla \times \vec{V}$, here $\vec{V} = \hat{e}_1 V_1 + \hat{e}_2 V_2 + \hat{e}_3 V_3$.

$$\boxed{\nabla \times \vec{V} = \frac{1}{h_1 h_2 3_3} \begin{vmatrix} h_1 \hat{e}_1 & h_2 \hat{e}_2 & h_3 \hat{e}_3 \\ \frac{\partial}{\partial u_1} & \frac{\partial}{\partial u_2} & \frac{\partial}{\partial u_3} \\ h_1 V_1 & h_2 V_2 & h_3 V_3 \end{vmatrix}} \quad \text{— (11)}$$

Take $\nabla \times \vec{V} = \nabla \times (V_1 \hat{e}_1 + V_2 \hat{e}_2 + V_3 \hat{e}_3)$

$$= \underbrace{\nabla \times (V_1 \hat{e}_1)}_{\text{consider only this component}} + \nabla \times (V_2 \hat{e}_2) + \nabla \times (V_3 \hat{e}_3) \quad \text{— (12)}$$

$\nabla \times (V_1 \hat{e}_1) = \nabla \times (V_1 h_1 \nabla \vec{u}_1)$

$= \nabla (V_1 h_1) \times \nabla \vec{u}_1 + V_1 h_1 \underbrace{\nabla \times \nabla \vec{u}_1}_{\text{zero}}$

{where we have used $\nabla \vec{u}_1 = \frac{\hat{e}_1}{h_1}$}

$= \nabla (V_1 h_1) \times \nabla \vec{u}_1$

$= \nabla (V_1 h_1) \times \frac{\hat{e}_1}{h_1}$

$= \left[\frac{\hat{e}_1}{h_1} \frac{\partial}{\partial u_1} (V_1 h_1) + \frac{\hat{e}_2}{h_2} \frac{\partial}{\partial u_2} (V_1 h_1) + \frac{\hat{e}_3}{h_3} \frac{\partial}{\partial u_3} (V_1 h_1) \right] \times \left(\frac{\hat{e}_1}{h_1} \right)$

$$\boxed{\nabla \times V_1 \hat{e}_1 = \frac{\hat{e}_2}{h_3 h_1} \frac{\partial}{\partial u_3} (V_1 h_1) - \frac{\hat{e}_3}{h_1 h_2} \frac{\partial}{\partial u_2} (V_1 h_1)} \quad \text{— (13)}$$

use Eq^n. 13 in Eq. 12

$$\nabla\times\vec{V} = \frac{\hat{e}_2}{h_3h_1}\frac{\partial}{\partial u_3}(V_1h_1) - \frac{\hat{e}_3}{h_1h_2}\frac{\partial}{\partial u_2}(V_1h_1)$$

$$+ \frac{\hat{e}_3}{h_1h_2}\frac{\partial}{\partial u_1}(V_2h_2) - \frac{\hat{e}_1}{h_2h_3}\frac{\partial}{\partial u_3}(V_2h_2)$$

$$+ \frac{\hat{e}_1}{h_2h_3}\frac{\partial}{\partial u_2}(V_3h_3) - \frac{\hat{e}_2}{h_3h_1}\frac{\partial}{\partial u_1}(V_3h_3)$$

$$= \frac{\hat{e}_1}{h_2h_3}\left[\frac{\partial}{\partial u_2}(V_3h_3) - \frac{\partial}{\partial u_3}(V_2h_2)\right]$$

$$+ \frac{\hat{e}_2}{h_3h_1}\left[\frac{\partial}{\partial u_3}(V_1h_1) - \frac{\partial}{\partial u_1}(V_3h_3)\right]$$

$$+ \frac{\hat{e}_3}{h_1h_2}\left[\frac{\partial}{\partial u_1}(V_2h_2) - \frac{\partial}{\partial u_2}(V_1h_1)\right]$$

$$\boxed{\text{or} \quad \nabla\times\vec{V} = \frac{1}{h_1h_2h_3}\begin{vmatrix} h_1\hat{e}_1 & h_2\hat{e}_2 & h_3\hat{e}_3 \\ \frac{\partial}{\partial u_1} & \frac{\partial}{\partial u_2} & \frac{\partial}{\partial u_3} \\ h_1V_1 & h_2V_2 & h_3V_3 \end{vmatrix}}$$